III. “Abstract of some Results with respect to Doubly Periodic Elliptic Functions of the Second and Third Kinds.”

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Received March 5, 1885.

Elliptic Functions of the Second Kind.

1. If $u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, $\mathrm{K}=\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, $\mathrm{K}'=\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}$, $k^2+k'^2=1$, $\theta=\mathrm{am}u$, $\sin\theta=\sin\mathrm{am}u$, $\cos\theta=\cos\mathrm{am}u$, $\sqrt{1-k^2\sin^2\theta}=\Delta\mathrm{am}u$, it is known that—

$$\sin\mathrm{am}(u+4\mathrm{K}+4i\mathrm{K}')=\sin\mathrm{am}u,$$

$$\cos\mathrm{am}(u+4\mathrm{K}+4i\mathrm{K}')=\cos\mathrm{am}u,$$

$$\Delta\mathrm{am}(u+4\mathrm{K}+4i\mathrm{K}')=\Delta\mathrm{am}u.$$

Similar results are true when u is replaced by the second elliptic integral $a=\int_0^\theta \sqrt{1-k^2\sin^2\theta}\,d\theta$, *and a corresponding change is made in Jacobi's notation.*

In fact, putting $\theta=\mathbf{am}a$, $\sin\theta=\sin\mathbf{am}a$, $\cos\theta=\cos\mathbf{am}a$, $\sqrt{1-k^2\sin^2\theta}=\Delta\mathbf{am}a$, we have the following theorems, viz.:—

$$\sin\mathbf{am}\{a+4\mathrm{E}+4i(\mathrm{K}'-\mathrm{E}')\}=\sin\mathbf{am}a,$$

$$\cos\mathbf{am}\{a+4\mathrm{E}+4i(\mathrm{K}'-\mathrm{E}')\}=\cos\mathbf{am}a,$$

$$\Delta\mathbf{am}\{a+4\mathrm{E}+4i(\mathrm{K}'-\mathrm{E}')\}=\Delta\mathbf{am}a,$$

if $\mathrm{E}=\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta$, $\mathrm{E}'=\int_0^{\frac{\pi}{2}} \sqrt{1-k'^2\sin^2\theta}\,d\theta$, $k^2+k^{2\prime}=1$, and K, K′ are the same quantities as above.

The fundamental equations satisfied by the new functions $\sin\mathbf{am}a$, $\cos\mathbf{am}a$, $\Delta\mathbf{am}a$ are—

(1.) $\sin\mathbf{am}\{a+b-k^2\sin\mathbf{am}a\sin\mathbf{am}bf(a,b)\}=f(a,b)$,

(2.) $\cos\mathbf{am}\{a+b-k^2\sin\mathbf{am}a\sin\mathbf{am}bf(a,b)\}=\phi(a,b)$,

(3.) $\Delta\mathbf{am}\{a+b-k^2\sin\mathbf{am}a\sin\mathbf{am}bf(a,b)\}=\psi(a,b)$,

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where $f(a,\ b)=\sin \mathrm{am}a \cos \mathrm{am}b \Delta \mathrm{am}b+\sin \mathrm{am}b \cos \mathrm{am}a \Delta \mathrm{am}a$, ÷

$$\phi(a,\ b)=\cos \mathrm{am}a \cos \mathrm{am}b-\sin \mathrm{am}a \Delta \mathrm{am}a \sin \mathrm{am}b \Delta \mathrm{am}b, \div$$

$$\psi(a,\ b)=\Delta \mathrm{am}a \Delta \mathrm{am}b-k^2 \sin \mathrm{am}a \cos \mathrm{am}a \sin \mathrm{am}b \cos \mathrm{am}b, \div$$

and common denominator $=1-k^2 \sin^2 \mathrm{am}a \sin^2 \mathrm{am}b$.

The theory of the two periods can be derived from the above and the relation

$$\int_0^{\sin^{-1}\frac{1}{k}} \sqrt{1-k^2 \sin^2 \phi}\, d\phi=$$

$$\int_0^{\frac{\pi}{2}} \sqrt{1-k^2 \sin^2 \phi}\, d\phi+i\left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k'^2 \sin^2 \theta}}-k^2 \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-k'^2 \sin^2 \theta)^{\frac{3}{2}}}\right\}$$

or $$\int_0^{\sin^{-1}\frac{1}{k}} \sqrt{1-k^2 \sin^2 \phi}\, d\phi=\mathrm{E}+i(\mathrm{K}'-\mathrm{E}'),$$

where $i=\sqrt{-1}$ and $\sin \phi=\dfrac{1}{\sqrt{1-k'^2 \sin^2 \theta}}$.

Elliptic Functions of the Third Kind.

2. Again, if u be replaced by $p=\displaystyle\int_0^{\theta} \frac{d\theta}{(1+n \sin^2 \theta) \sqrt{1-k^2 \sin^2 \theta}}$ and θ be taken $=\mathrm{am}p$, $\sin \theta=\sin \mathrm{am}p$, $\cos \theta=\cos \mathrm{am}p$, $\sqrt{1-k^2 \sin^2 \theta}=\Delta \mathrm{am}p$, we have

$$\sin \mathrm{am}\{p+4\Pi+4i\mathrm{P}'\}=\sin \mathrm{am}p\ ;$$

$$\cos \mathrm{am}\{p+4\Pi+4i\mathrm{P}'\}=\cos \mathrm{am}p\ ;$$

$$\Delta \mathrm{am}\{p+4\Pi+4i\mathrm{P}'\}=\Delta \mathrm{am}p\ ;$$

where $$\Pi=\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+n \sin^2 \theta) \sqrt{1-k^2 \sin^2 \theta}},$$

$$\Pi'=\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+n' \sin^2 \theta) \sqrt{1-k'^2 \sin^2 \theta}},$$

$$k^2+k'^2=1,\quad n'(1+n)=-k'^2,\quad \mathrm{P}'=\mathrm{K}'-\frac{n}{1+n}\Pi'.$$

In this case the periods depend upon the relation

$$\int_0^{\sin^{-1}\frac{1}{k}} \frac{d\phi}{(1+n \sin^2 \phi) \sqrt{1-k^2 \sin^2 \phi}}=\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1+n \sin^2 \phi) \sqrt{1-k^2 \sin^2 \phi}}$$

$$+i\left\{\int_0^{\pi}\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}-\frac{n}{1+n}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\left(1-\frac{k'^2}{1+n}\sin^2\theta\right)\sqrt{1-k'^2\sin^2\theta}}\right\}$$

where $\sin\phi=\frac{1}{\sqrt{1-k'^2\sin^2\theta}}$.

3. Remarks on a $\zeta(u)$ function, which includes Jacobi's $Z(u)$ as a particular case.

If we regard p as a function of u and write $\frac{P'}{\Pi}\div\frac{K'}{K}=\mu$, there seems to be a $\zeta(u)$ relation of the form $\zeta(u)=\frac{\pi}{2(1-\mu)K'}\left\{\frac{p}{\Pi}-\frac{u}{K}\right\}$.

This satisfies the two equations

$$\zeta(u+2K)=\zeta(u),$$

$\zeta(u+2iK')-\zeta(u)=-\frac{\pi i}{K}$, as may be seen by changing (1) u, p into $u+2K$, $p+2\Pi$; (2) u, p into $u+2iK'$, $p+2iP'$, respectively.

It is easy to show by ordinary trigonometrical formulæ that we get a solution of these functional equations by putting

$$\zeta(u)=\frac{2\pi}{u_0K}\left\{\frac{q}{1-q^2}\sin u_0\sin\frac{\pi u}{K}+\tfrac{1}{2}\frac{q^2}{1-q^4}\sin 2u_0\sin\frac{2\pi u}{K}\right.$$

$$\left.+\tfrac{1}{3}\frac{q^3}{1-q^6}\sin 3u_0\sin\frac{3\pi u}{K}+\quad.\;.\;.\;.\;.\quad \textit{ad infin.}\right\},$$

where u_0 is a constant and $q=e^{-\pi\frac{K}{K}}$.

The deduction of—

$$a-\frac{E}{K}u=\frac{2\pi}{K}\left\{\frac{q}{1-q^2}\sin\frac{\pi u}{K}+\frac{q^2}{1-q^4}\sin\frac{2\pi u}{K}.\quad.\;.\;.\right\}$$

presents no particular difficulty.

This is Jacob's $Z(u)$.

The above considerations suggest the very interesting question whether we may not take u to be a function of p, and so write

$$p-\frac{\Pi}{K}u=A_1\sin\frac{\pi p}{\Pi}+A_2\sin\frac{2\pi p}{\Pi}+A_3\sin\frac{3\pi p}{\Pi}+.\quad.\;.\quad \textit{ad infin.}$$

Primâ facie this does not seem to be an impossible equation, since the series can be reversed by Lagrange's theorem, *i.e.*, we may have

$$p-\frac{\Pi}{K}u=B_1\sin\frac{\pi u}{K}+B_2\sin\frac{2\pi u}{K}+.\quad.\;.\;.$$

If the assumption in question is valid, there follows the relation

$$1-\frac{\Pi}{K}(1+n\sin^2 \mathrm{am}p)=\frac{\pi}{\Pi}\left\{A_1\cos\frac{\pi p}{\Pi}+2A_2\cos\frac{2\pi p}{\Pi}+\ .\ \ .\ \ .\right\},$$

or, in other words, the elliptic functions of the third kind, sin $\mathrm{am}p$, &c., can be expressed by harmonic series involving multiples of $\frac{p}{\Pi}$.

March 19, 1885.

THE TREASURER in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "On 'Transfer-resistance' in Electrolytic and Voltaic Cells." By G. GORE, LL.D., F.R.S. Received March 2, 1885.

(Abstract.)

The existence of this phenomenon has been a matter of doubt ever since the year 1831, and the question has been examined by many investigators. In the present paper are described a series of methods by means of which its reality has been determined. Other methods are given for measuring the amounts of such "resistance," either collectively at the two electrodes of an electrolytic cell, or separately at each electrode. Modes of obviating the interference of polarisation, and of securing success in the measurements, are also described.

The influence of various circumstances upon the phenomenon were investigated, viz., strength and density of current; total resistance; density of current and size of electrode; composition of the electrolyte; strength of ditto; combined electrolytic cells; temperature; and chemical corrosion. The relations of the phenomenon to size of plate in voltaic cells, to the positive and negative plates respectively, and to strength of current in those cells, were also examined, and the results are given.

The following are the chief facts established by this research:— That a species of electric "resistance," distinct from that of polarisation and of ordinary conduction-resistance, varying greatly in amount in different cases, exists at the surfaces of mutual contact of metals and liquids in electrolytic and voltaic cells. That this "resistance"